Cycle and Path Decompositions of Even Hypercubes

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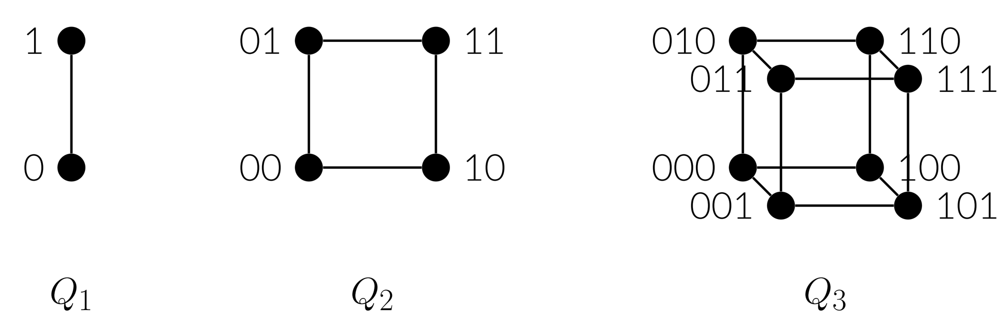
Hypercube graphs

For the n-dimensional hypercube Q_n:

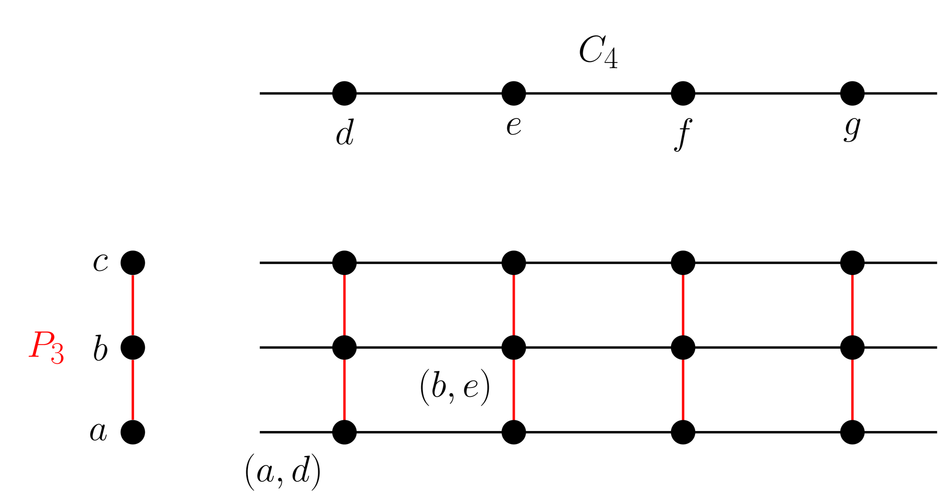
- $V(Q_n) = \{0,1\}^n$
- $E(Q_n) =$ pairs of vertices that differ in one coordinate



Cartesian products of graphs

- $V(G \square H) = \{V(G) \times V(H)\}$
- $E(G \square H) = \{(u,v)(u',v') \mid u = u', vv' \in E(H) \text{ or } uu' \in V(G), v = v'\}$

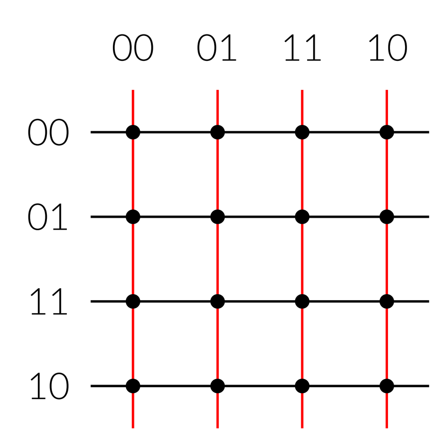
Example: $P_3 \square C_4$



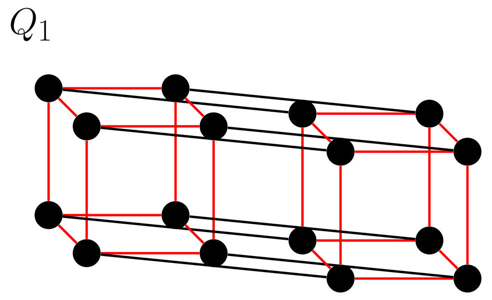
Hypercube graphs are Cartesian products:

$$Q_{n+m} = Q_n \square Q_m$$

Example: $Q_4 = Q_2 \square Q_2 = C_4 \square C_4$



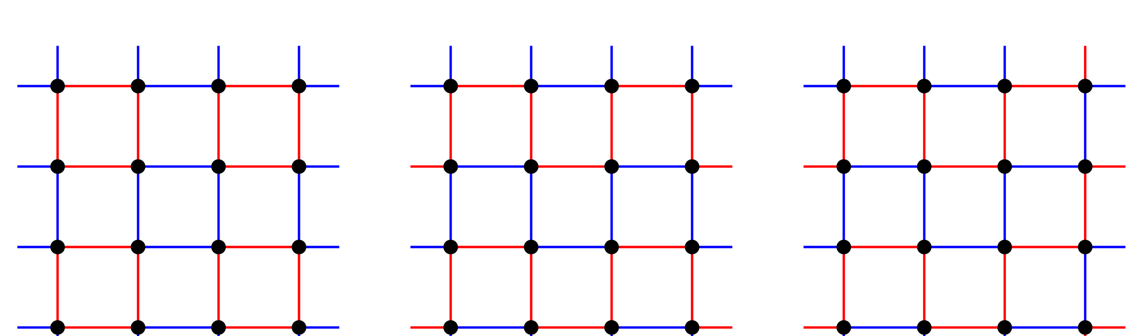
Example: $Q_4 = Q_3 \square Q_1$



Graph decompositions

A **decomposition** of a graph G is a set of edge-disjoint subgraphs whose edges partition $E(G)$.

Example: Decompositions of Q_4 into $8C_4$'s, $4C_8$'s, and $2C_{16}$'s.



Main Research Question

What graphs can decompose the hypercube $\mathbf{Q_n}$?

Our research: Which paths and cycles decompose even-dimensional hypercubes?

Necessary conditions for decompositions

Necessary conditions for a graph G to decompose the hypercube Q_n:

- $|V(G)| \le |V(Q_n)| = 2^n$
- $|E(G)|$ divides $|E(Q_n)| = n2^{n-1}$
- The gcd of the degrees of the vertices of G divides n
- The gcd of the number of edges in each direction of G divides 2^{n-1}

Prior results

1. Paths in Q_n for odd n [3]
2. Hamiltonian cycles in Q_n for even n [1]
3. Cycles of length less than $2^n/n$ in Q_n for even n [2]
4. Cycles of length 2^i for $1 < i < n$ in Q_n for even n [4]

New cycle decompositions

We found new decompositions of hypercubes into long cycles whose lengths have odd divisors.

Theorem: If n is even and the sum of at most 3 powers of 2, then the cycle with the largest length divisible by n while still satisfying the necessary conditions decomposes Q_n.

Examples:

- Q_{14} can be decomposed into eight cycles of length $14 \cdot 2^{10}$. This is the longest cycle whose length is a multiple of 14 that satisfies the necessary conditions. This theorem applies since $14 = 2^3 + 2^2 + 2^1$.
- Q_{22} can be decomposed into sixteen cycles of length $22 \cdot 2^{17}$. This theorem applies because $22 = 2^4 + 2^2 + 2^1$.
- Q_{28} can be decomposed into sixteen cycles of length $28 \cdot 2^{23}$. This theorem applies because $28 = 2^4 + 2^3 + 2^2$.

Decompositions using generalized hypercubes

A **generalized hypercube**, denoted C_{2k}^n, is the n-fold Cartesian product of cycle graphs C_{2k}, where $k \ge 2$.

Example: $C_{16}^3 = C_{16} \square C_{16} \square C_{16}$

Theorem: A generalized hypercube C_x^{yz} can be decomposed into y copies of $(C_{xy})^z$.

Example: A decomposition of Q_{pq} into cycles whose length is divisible by p but not pq.

- Since $Q_{18} = Q_2^9 = C_4^9 = ((C_4)^3)^3$, it can be decomposed into three copies of the generalized hypercube $(C_{4^3})^3$. Each of these decomposes into four cycles of length $3 \cdot 2^{16}$. Thus Q_{z18} has a decomposition into $3 \cdot 4 = 12$ cycles of length $3 \cdot 2^{16}$.

Example: A decomposition of Q_{4n} into cycles.

- Since $Q_{12} = Q_4^3$, and Q_4 can be decomposed into two copies of C_{16}, Q_{12} can be represented as $(2C_{16})^3$. Since each copy of C_{16} is spanning in Q_4, Q_{12} can be decomposed into two copies of C_{16}^3. Each of these generalized hypercubes can be decomposed into four cycles of length $6 \cdot 2^9$. Hence, Q_{12} has a decomposition into $2 \cdot 4 = 8$ cycles of length $6 \cdot 2^9$.

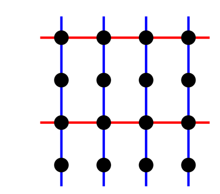
Anchored products and decompositions

Anchored Products

Given graphs G and H, with $X \subseteq V(G)$ and $Y \subseteq V(H)$ then the **anchored product** $(G,X) \boxplus (H,Y)$ is the graph where

- $V((G,X) \boxplus (H,Y)) = \{V(G) \times Y \cup X \times V(H)\}$
- $E((G,X) \boxplus (H,Y)) = \{(u,v)(u',v') \mid u = u', vv' \in E(H) \text{ or } uu' \in V(G), v = v' \in Y\}$

A **anchored torus** is an anchored product where G and H are cycles.

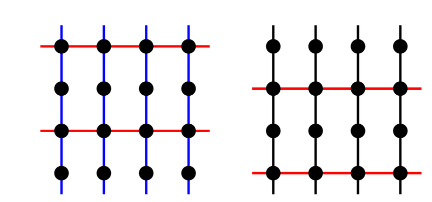


Anchored Decompositions

An **anchored decomposition** of a graph G is a set of ordered pairs $\{(G_1,V_1),\dots,(G_m,V_m)\}$ such that $\{G_1,\dots,G_m\}$ forms a decomposition of G, and $V_1,\dots,V_m$ is a partition of $V(G)$.

Theorem: Given an anchored decomposition of a graph G into m cycles of some length x, $G \square C_y$ has an anchored decomposition into m anchored tori.

Example: If G can be decomposed into two cycles C^1 and C^2, then the Cartesian product $G \square C$ can be decomposed into two anchored products $(C^1,V) \boxplus C$ and $(C^2,W) \boxplus C$, where V and W partition the vertices of G.



Decompositions of hypercubes using anchored products

We can use a decomposition of a hypercube Q_{2n} into anchored tori as an intermediate step to decompose Q_{2n} into cycles.

Example: Since $Q_6 = Q_4 \square Q_2$, and Q_4 can be decomposed into two copies of C_{16}, we can represent Q_6 as $(2C_{16}) \square C_4$. Thus Q_6 has a decomposition into two anchored tori of the form $(C_{16},V) \boxplus C_4$. Each of these anchored tori can then be decomposed into two cycles, resulting in a decomposition of Q_6 into 4 cycles.

Alternating Anchored Circuits

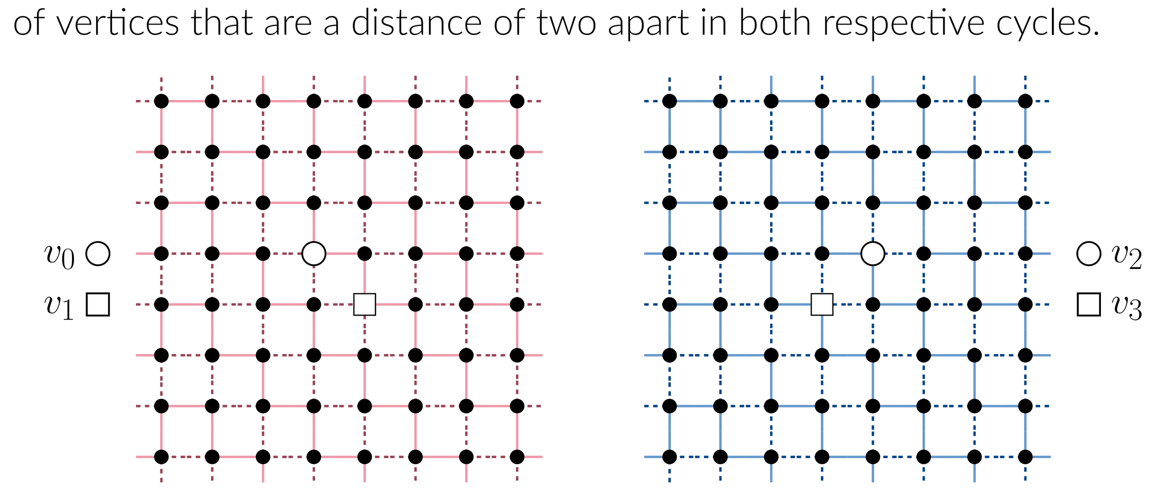
An **alternating anchored circuit** (C,V) is an anchored circuit such that

- If $v \in V$ then v has degree two in the circuit.
- The edges of C are 2-colored such that
  - Consecutive edges have the same color unless they share a vertex in V.
  - Each vertex is incident with at most two edges of each color.

Theorem: The anchored product of an alternating anchored circuit with a cycle can be decomposed into two cycles of the same length.

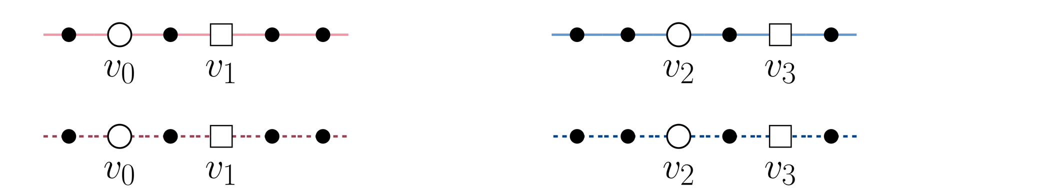
Example: Decomposing Q_{14} into cycles of length $14 \cdot 2^{10}$

$Q_{14} = Q_8 \square Q_4 \square Q_2$. Considering just $Q_8 = Q_4 \square Q_4$, we can decompose Q_8 into a pair of generalized hypercubes $2C_{16} \square 2C_{16} = 2C_{16}^2$. We can take an anchored decomposition of C_{16}^2 to decompose each copy into two anchored cycles. Furthermore, in each graph we pick a different parity pair of vertices that are a distance of two apart in both respective cycles.

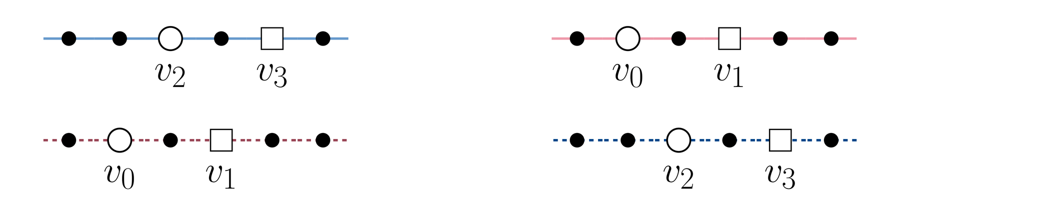


We can now consider all four of these anchored cycles independently.

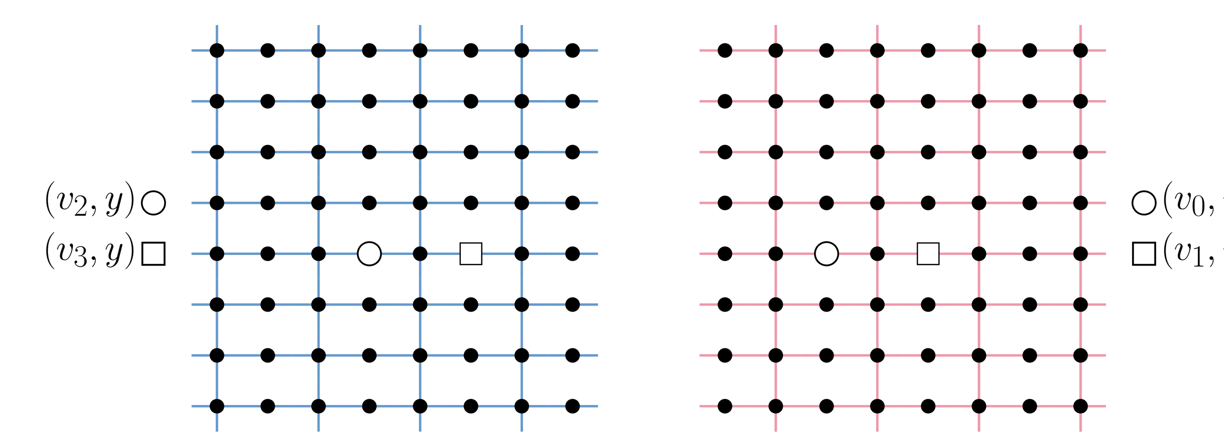
Decomposition of Q_{14} Continued



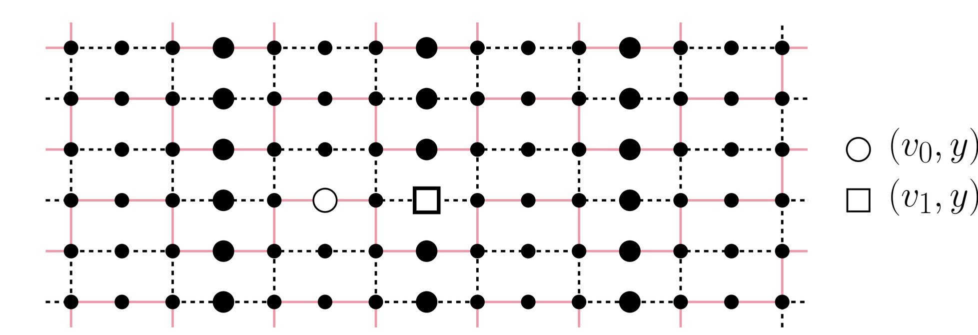
We can then swap one of the cycles from each pair.



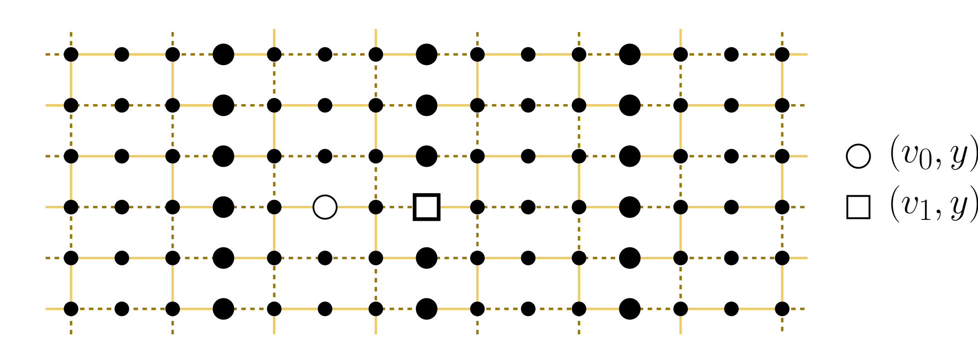
We can then take the Cartesian product of Q_8 with Q_4. Since Q_4 can be decomposed into two spanning copies of C_{16}, Q_{12} can be decomposed into four anchored tori using each of the four cycles above. Here is one of the tori from each pair.



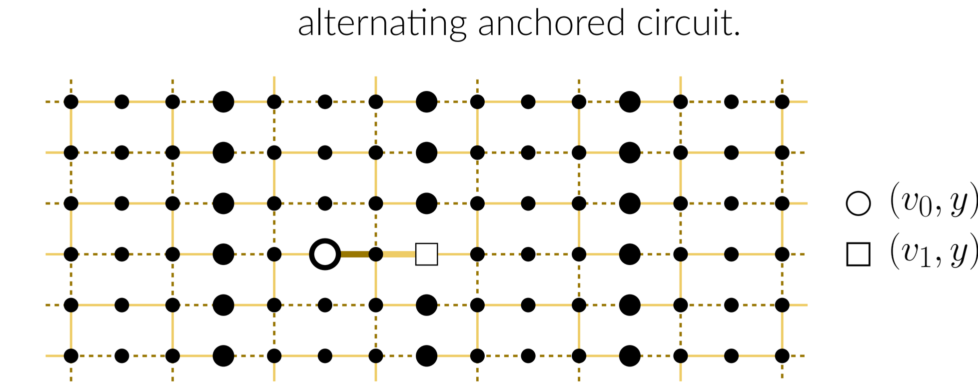
Considering just one of these anchored tori, (C,V), take the anchored decomposition into two cycles (bold vertices are the vertices of V).



Color each of these cycles in paths of alternating color satisfying properties of alternating anchored circuits.



Swap two edges in the anchored torus to connect the two cycles into an alternating anchored circuit.



We obtain an anchored decomposition of Q_{12} into 4 alternating anchored circuits. Taking the anchored product of each one with Q_2 we get a decomposition of Q_{14} into 8 cycles.

References

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[3] J. Erde, Decomposing the cube into paths, Discrete Mathematics 336 (2014) 41–45

[4] S. Gibson and D. Offner, Decompositions of even hypercubes into cycles whose length is a power of two, arXiv (2021) 2107.07450